Soliton stability in some knot soliton models

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Abstract

We study the issue of stability of static soliton-like solutions in some non-linear field theories which allow for knotted field configurations. Concretely, we investigate the AFZ model, based on a Lagrangian quartic in first derivatives with infinitely many conserved currents, for which infinitely many soliton solutions are known analytically. For this model we find that sectors with different (integer) topological charge (Hopf index) are *not* separated by an infinite energy barrier. Further, if variations which change the topological charge are allowed, then the static solutions are not even critical points of the energy functional. We also explain why soliton solutions can exist at all, in spite of these facts. In addition, we briefly discuss the Nicole model, which is based on a sigma-model type Lagrangian. For the Nicole model we find that different topological sectors are separated by an infinite energy barrier.

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1 Introduction

Nonlinear field theories, which allow for static soliton solutions of their field equations, have been studied intensively for many years due to their importance in a variety of fields in theoretical and mathematical physics. Indeed, applications range from elementary particle theory, where they may serve as effective field theories providing particle-like solutions, to condensed matter and solid state physics. One important criterion in the classification of these nonlinear field theories is given by the dimension of the base space (i.e., space-time) on which these fields exist. In the case of one space and one time dimension, static solutions have to solve a nonlinear ODE, which is usually much easier to solve than the nonlinear PDEs which result for solitons in higher dimensions. In addition, in $1+1$ dimensions many rigorous results on nonlinear field theories and their soliton solutions are known, and a vast mathematical apparatus for the analysis of these models has been developed. Among those are inverse scattering, integrability, and the zero curvature representation, where the latter generalizes to field theory the Lax pair representation of integrable systems in $0+1$ dimensions (that is, systems with finitely many degrees of freedom).

In higher dimensions (i.e., in $d+1$ dimensional space-time for $d>1$), on the other hand, much less is known about nonlinear field theories and their solutions. A generalization of the zero curvature representation to higher dimensions was proposed in [1], and it was shown there that this proposal leads to nonlinear field theories with infinitely many conservation laws, realizing thereby the concept of integrability in higher dimensions in a concrete and well-defined manner. Besides this, some specific nonlinear field theories in higher dimensions have been studied with the intention of more phenomenological applications. As far as static finite-energy solutions (solitons) are relavant in these applications, the selected nonlinear theories should, of course, really support such static solutions. One necessary condition for the existence of static finite-energy solutions is provided by the Derrick scaling argument (see, e.g., [2, 3]). The Derrick scaling argument simply says that static solutions with finite, nonzero energy cannot exist if it is possible for arbitrary field configurations with finite, nonzero energy to rescale the energy functional to arbitrarily small values by a base space scale transformation $\mathbf{r} \rightarrow \mu\mathbf{r}$, where $\mathbf{r} \in \mathbb{R}^d$ are the base space coordinates and μ, with $0 < \mu < \infty$, is the scale parameter. Let us be more precise for the class of theories we are interested in. We assume that we have only scalar fields Φ_a, $a = 1 \ldots N$, which transform trivially under the above scale transformation, $\Phi_a \to \Phi_a$, and that the energy density for static configurations is a sum of terms each of which is homogeneous (of degree h) in space derivatives $\nabla_k \Phi_a$. Then each such term $\mathcal{E}_h(\Phi_a, \nabla_k \Phi_a)$ contributes a term $E_h[\Phi_a] = \int d^d\mathbf{r} \mathcal{E}_h(\Phi_a, \nabla_k \Phi_a)$ to the energy functional, where $E_h[\Phi_a]$ transforms as $E_h[\Phi_a] \to \mu^{d-h} E_h[\Phi_a]$ under the scale transformation. There are now several possibilities to obey the Derrick criterion for the possible existence of solutions. If the energy density only consists of one term, then necessarily $d = h$, that is, the degree of homogeneity in first derivatives, h, must be equal to the dimension of space, d. If the energy density consists of several terms, then at least one term must obey $d - h > 0$, and at least one further term must have the opposite behaviour, $d - h < 0$ (unless all terms obey the scale invariance condition $d = h$). All our concrete investigations will be for $3 + 1$ dimensional space-time, therefore we assume $d = 3$ from now on.

One well-known example of a field theory in $d = 3$ dimensions is the Skyrme model [4], which has applications in nuclear physics. It has the group SU(2) as field configuration space (target space),[1] and the fields are supposed to describe pions, whereas the soliton solutions are related to the nucleons. The energy functional of the Skyrme model consists of a sigma-model type term quadratic in first derivatives, with $d - h = 3 - 2 = 1$, and of a further term quartic in derivatives (the "Skyrme term") with $d - h = 3 - 4 = -1$, so it obeys the Derrick criterion for the possibility of static solutions. Occasionally a third "potential" term is added to the Lagrangian density (and, consequently, to the energy density), which only depends on the fields, but not on derivatives. This term is usually assumed to make the pions massive. Soliton solutions have been found numerically for the Skyrme model both with and without the pion mass term [6, 7, 8].

Another well-known model with applications both to field theory and condensed matter physics is the Faddeev–Niemi model ([9], [10]). This model may be derived from the Skyrme model by simply restricting its target space from SU(2) $\sim S^3$ to the two-sphere S^2 (e.g., the equator of the Skyrme S^3, or SU(2)/U(1)). The target space of the Faddeev–Niemi model is two-dimensional, therefore the solitons are now line-like instead of point-like (where the soliton position may be defined, e.g., by the loci where the fields

[1] In particle physics, SU(3) has to be used, see e.g. [5].

take values which are antipodal to their vacuum value). More precisely, the solitons of the Fadeev–Niemi model are links or knots. The corresponding topological index classifying a finite energy field configuration is a linking number (the Hopf index), whereas it is a winding number for the Skyrme model. The existence of soliton solutions in the Faddeev–Niemi model has been proven in [11], and confirmed by numerical calculations, e.g., in ([12] – [15]).

The Faddeev–Niemi model has the Lagrangian density

$$\mathcal{L} = \frac{m^2}{2}\mathcal{L}_2 - \lambda\mathcal{L}_4 + \mathcal{L}_0 \tag{1}$$

where m is a constant with dimension of mass, λ is a dimensionless coupling constant, $\mathcal{L}_2$ is

$$\mathcal{L}_2 = \frac{\partial_\mu u\,\partial^\mu \bar{u}}{(1+u\bar{u})^2}, \tag{2}$$

and $\mathcal{L}_4$ is

$$\mathcal{L}_4 = \frac{(\partial^\mu u\,\partial_\mu \bar{u})^2 - (\partial^\mu u\,\partial_\mu u)(\partial^\nu \bar{u}\,\partial_\nu \bar{u})}{(1+u\bar{u})^4}. \tag{3}$$

Further, u is a complex field which parametrizes the stereographic projection of the target S^2, see Section 2 below for more details. Again we have allowed for a pure potential term $\mathcal{L}_0(u,\bar{u})$. This term is not required by the Derrick criterion, but it may be necessary for some applications of the model (e.g., by providing a symmetry breaking of the target space symmetry group from SU(2) to U(1)).

It is also possible to select Lagrangian densities which consist of one term only and are constructed from $\mathcal{L}_4$ or $\mathcal{L}_2$ exclusively, and obey the Derrick criterion $h = 3$. They are, however, necessarily non-polynomial. For $\mathcal{L}_4$ the appropriate choice is

$$\mathcal{L}_{\mathrm{AFZ}} = -(\mathcal{L}_4)^{\frac{3}{4}}. \tag{4}$$

This model has been introduced and studied by Aratyn, Ferreira and Zimerman (AFZ) in [16], [17]. AFZ found infinitely many analytic soliton solutions for this model by using an ansatz with toroidal coordinates. The analysis of the AFZ model was carried further in ([18]), where, among other results, all the space-time and (geometric) target space symmetries of the AFZ model were determined, and, further, the use of the ansatz with toroidal coordinates was related to the conformal symmetry of the model (more precisely, of the static equations of motion). It turns out that the AFZ model has infinitely many target space symmetries and, thus, infinitely many conservation laws. Moreover, it also shows classical integrability in a different sense, because the static field equations resulting from the ansatz with toroidal coordinates may be solved by simple integration.

The other model, based solely on $\mathcal{L}_2$, is the Nicole model

$$\mathcal{L}_{\mathrm{Ni}} = (\mathcal{L}_2)^{\frac{3}{2}}. \tag{5}$$

This model has first been proposed by Nicole ([19]), and it was shown in the same paper that the simplest Hopf map with Hopf index 1 is a soliton solution for this model. (This model is, in fact, the restriction to S^2 target space of a non-polynomial $SU(2)$ model which was studied first in [20] as a possible candidate for a pion model.) The Nicole model shares the conformal symmetry with the AFZ model and, therefore, the ansatz with toroidal coordinates may be used again to simplify the static field equations (to reduce them to an ordinary differential equation). However, the Nicole model only has the obvious symmetries - the conformal base space symmetries (in the static case) and the modular target space symmetries, see [21]. Consequently, the field equations are no longer integrable, and the solutions are no longer available in closed, analytic form (except for the simplest case with Hopf index one[2]). For a detailed investigation of soliton solutions within the ansatz in toroidal coordinates we refer to [23].

It is the main purpose of this paper to study stability issues of these two latter models with explicit solutions. Firstly, let us emphasize that the Derrick criterion is a necessary condition for the existence of solutions, but by no means a sufficient one. So both the existence and the stability of soliton solutions have to be investigated independently. We find the more surprising results for the AFZ model.[3] In this model it turns out that there exists a symmetry transformation (which maps solutions into solutions) which connects an arbitrary solution to the trivial vacuum solution $u = 0$. Consequently, soliton solutions with a nonzero topological index are not separated from the vacuum by an energy barrier. Further, soliton solutions are not even critical points of the energy functional, because there exists a nonzero first variation

[2]Exact solitons with higher Hopf index have been found in modified Nicole models [22]

[3]Warnings about its stability, based on the base space scale invariance $\mathbf{r} \to \mu \mathbf{r}$ of the model, have been given in [3].

of the energy functional into the direction of the symmetry transformation. This immediately poses an apparent paradox: (static) soliton solutions solve the variational equations of the energy functional, therefore they should be critical points of the energy functional by definition. If, on the other hand, the above-mentioned symmetry implies that there exists a non-flat direction of the energy functional for arbitrary fields, this seems to imply that finite energy solutions cannot exist, much like the original Derrick scaling argument. Static soliton solutions of the AFZ model do exist, however, and have been constructed in ([16], [17]). In this paper, therefore, we not only show the existence of the symmetry transformation connecting solutions to the vacuum, we also explain how the above-mentioned apparent paradox is resolved, by considering inherent source terms.

In Section 2 we study the reduced energy functional which follows from an ansatz using toroidal coordinates. In Subsection 2.1 we perform this study for a simple toy model. The toy model has the same stability problems as the AFZ model, but much simpler field equations, therefore we mainly introduce it for illustrative purposes. This toy model may, however, be of some independent interest. It consists of a non-quadratic kinetic energy term, and such models have been studied in the context of "fake inflation" [24] (where the change in light propagation caused by the non-conventional kinetic term mimics a nontrivial geometry, as far as the propagation of light is concerned), and in the context of the so-called modified Newtonian dynamics (MOND), see, e.g., [25]. In addition, the toy model is equivalent to the electrostatic sector of a nonlinear theory of electrodynamics. This issue we discuss briefly in Subsection 3.1. In Subsection 2.2, at first we discuss the boundary conditions implied by the Hopf index for the ansatz within toroidal coordinates. Then we study the reduced energy functional of the AFZ model, using the analogy with the toy model. We establish the instability and show that the existence of solutions which are not critical points can be attributed to boundary term contributions to the reduced energy functional. In Section 2.3 we briefly study the reduced energy functional of the Nicole model. We find that in this case different topological sectors are indeed separated by an infinite energy barrier, as one generally expects for topological soliton models.

In Section 3 we study the full energy functional. In Subsection 3.1 we again investigate the toy model. We show that the apparent paradox is resolved by the presence of singular, delta-function like flux terms in a conservation equation. Further, we briefly introduce the above-mentioned nonlinear electrodynamics. In Subsection 3.2 we investigate the AFZ model. It turns out that the apparent paradox related with the existence of solutions is resolved in this case by the existence of an inhomogeneous, singular flux term in just one of the infinitely many conservation equations of the model.

Section 4 is devoted to the discussion of our results, emphasizing both the stability problems and the resolution of the apparent paradox. Further, we discuss possible generalizations of our results, and their application to and implications for further models. In Appendix A we calculate the Hopf index for a field within the ansatz in toroidal coordinates.

2 The reduced energy functional

2.1 A toy model

Firstly, let us introduce a simple toy model which already shows the essential features we want to study in the sequel. We choose the energy functional

$$E[\Phi] = \int d^3\mathbf{r} (\nabla\Phi \cdot \nabla\Phi)^{\frac{3}{2}} \tag{6}$$

where Φ is a real scalar field. Here the non-integer power of $\frac{3}{2}$ is chosen precisely as to make the energy functional invariant under a base space scale transformation $\mathbf{r} \rightarrow \mu\mathbf{r}$, avoiding thereby the usual Derrick scaling instability. There is, however, another scale transformation which does *not* leave invariant the energy functional. In fact, as both the energy functional and the resulting Euler–Lagrange equations are homogeneous in the scalar field Φ, the target space scale transformation $\Phi \rightarrow \lambda\Phi$ is a symmetry of the Euler–Lagrange equations which does not leave invariant the energy. As long as there exists no normalization condition for Φ which fixes the value of the scale factor λ (and enters the energy functional, e.g., via a Lagrange multiplier), the same Derrick type scaling argument applies and seems to prevent the existence of finite energy solutions to the Euler–Lagrange equations. Nevertheless, finite energy solutions exist, as we want to show now. We introduce toroidal coordinates (η, ξ, φ), $\eta \in [0, \infty]$, $\xi, \varphi \in [0, 2\pi]$, via

$$\begin{aligned} x &= q^{-1} \sinh\eta \cos\varphi \ , \quad y = q^{-1} \sinh\eta \sin\varphi \\ z &= q^{-1} \sin\xi \quad ; \qquad q = \cosh\eta - \cos\xi \end{aligned} \tag{7}$$

in order to reduce the Euler–Lagrange equations to an ordinary differential equation (ODE). Further, we need the volume form

$$dV \equiv d^3r = q^{-3} \sinh\eta \, d\eta \, d\xi \, d\varphi \tag{8}$$

and the gradient

$$\nabla = (\nabla\eta)\partial_\eta + (\nabla\xi)\partial_\xi + (\nabla\varphi)\partial_\varphi = q(\hat{e}_\eta\partial_\eta + \hat{e}_\xi\partial_\xi + \frac{1}{\sinh\eta}\hat{e}_\varphi\partial_\varphi) \tag{9}$$

where $(\hat{e}_\eta, \hat{e}_\xi, \hat{e}_\varphi)$ form an orthonormal frame in $\mathbb{R}^3$. Assuming now that the scalar field Φ only depends on η, the energy functional simplifies to

$$E[\Phi] = 4\pi^2 \int_0^\infty d\eta \sinh\eta |\Phi_\eta|^3 \tag{10}$$

where $\Phi_\eta \equiv \partial_\eta \Phi$.

Remark: Due to the conformal base space symmetry of the Euler–Lagrange equations, these equations are compatible with the ansatz $\Phi = \Phi(\eta)$. It then follows from the principle of symmetric criticality that the resulting ODE is identical to the Euler–Lagrange equation which is derived from the reduced energy functional (10), see, e.g., [2, 3].

We prefer to introduce the new variable

$$t = \sinh\eta \tag{11}$$

which leads to the energy functional

$$E[\Phi] = 4\pi^2 \int_0^\infty dt t(1+t^2)|\Phi_t|^3. \tag{12}$$

We now skip the absolute value signs by assuming that $\Phi_t \geq 0 \, \forall \, t$, so that an infinitesimal variation of the scalar field leads to

$$\begin{aligned} \delta E &\equiv E[\Phi + \delta\Phi] - E[\Phi] \\ &\simeq 12\pi^2 \int_0^\infty dt t(1+t^2)\Phi_t^2 \delta\Phi_t \\ &= 12\pi^2 \left[t(1+t^2)\Phi_t^2 \delta\Phi \right]_0^\infty - \qquad (13) \\ &\quad\; 12\pi^2 \int_0^\infty dt \delta\Phi \frac{d}{dt}[t(1+t^2)\Phi_t^2] \qquad (14) \end{aligned}$$

and, therefore, to the Euler–Lagrange equation

$$\frac{d}{dt}[t(1+t^2)\Phi_t^2] = 0 \tag{15}$$

with the solution

$$\Phi_t = C[t(1+t^2)]^{-\frac{1}{2}} \tag{16}$$

where C is an integration constant. For $C > 0$ it holds indeed that $\Phi_t \geq 0$. The scalar field itself,

$$\Phi(t) = C\int_0^t \frac{dt'}{\sqrt{t'(1+t'^2)}} + c', \tag{17}$$

can be expressed in terms of elliptic functions, but we do not need the explicit expression here. The energy of this field configuration is finite

$$E = 4\pi^2 C^3 \int_0^\infty \frac{dt}{\sqrt{t(1+t^2)}} = 32C^3\pi^{\frac{3}{2}}\Gamma^2(\frac{5}{4}) \tag{18}$$

(where $\Gamma(\cdot)$ is the Gamma function) and can take any positive value due to the arbitrary integration constant C. Further, for this field configuration the variation in the energy steming from the boundary term (13) is

$$\delta E = 12\pi^2 C^2[\delta\Phi(\infty) - \delta\Phi(0)]. \tag{19}$$

Specifically, for a variation proportional to the field, $\delta\Phi = \epsilon\Phi$, this is nonzero,

$$\delta E = 12\pi^2 C^2\epsilon[\Phi(\infty) - \Phi(0)] = 96C^2\epsilon\pi^{\frac{3}{2}}\Gamma^2(\frac{5}{4}). \tag{20}$$

In short, there exist finite energy solutions to the Euler–Lagrange equations in spite of the scaling instability under $\Phi \to \lambda\Phi$, because such a variation $\delta\Phi = \epsilon\Phi$ produces a nonzero contribution to δE from the boundary term. It follows that the solution (16), despite being of finite energy, is *not* a critical point of the energy functional. The instability is in agreement with more general mathematical criteria for models without topological charges obtained by Rubakov and others, see, e.g., page 58 of Ref. [2].

2.2 The AFZ model

The degrees of freedom for both the AFZ and the Nicole model are given by a three-component unit vector field, that is, a map

$$\vec{n}(\mathbf{r}) \, : \, \mathrm{I\!R}^3 \to S^2 \, , \quad \vec{n}^2 = 1 \tag{21}$$

where the tip of the unit vector field spans the unit two-sphere, or via stereographic projection

$$\vec{n} = \frac{1}{1+\mid u \mid^2} \left(u + \bar{u}, -i(u - \bar{u}), 1 - u\bar{u} \right) \, ; \qquad u = \frac{n_1 + in_2}{1 + n_3}. \tag{22}$$

by a complex field

$$u(\mathbf{r}) \, : \, \mathrm{I\!R}^3 \to \mathbb{C}_0 \tag{23}$$

where our conventions are such that the projection is from the south pole to the equatorial plane of the two-sphere. Further, $\mathbb{C}_0$ is the one-point compactified complex plane.

Topological solitons, i.e., static finite energy solutions with a non-zero integer value of the corresponding topological charge (the Hopf index) have been found for both models, so let us briefly describe the corresponding topological map (Hopf map). A Hopf map is a map $\phi : S^3 \to S^2$ or, via the stereographic projection from S^3 to one-point compactified $\mathrm{I\!R}_0^3$, a map $\phi : \mathrm{I\!R}_0^3 \to S^2$, and is characterized by the integer Hopf index (for details we refer to the Appendix A of Ref. [23]). Therefore, the above fields $\vec{n}$ and u can be identified with Hopf maps provided that the following two conditions hold. Firstly, the fields have to obey

$$\lim_{|\mathbf{r}|\to\infty} \vec{n} = \vec{n}_0 = \text{const} \, , \qquad \lim_{|\mathbf{r}|\to\infty} u = u_0 = \text{const} \tag{24}$$

so that they are, in fact, defined on one-point compactified Euclidean three-dimensional space $\mathrm{I\!R}_0^3$. Secondly, the fields have to take values in the full target spaces S^2 and $\mathbb{C}_0$, respectively, in such a way that the preimage of each point in target space is a closed line in the base space $\mathrm{I\!R}_0^3$.

Both the AFZ and the Nicole model have a conformally invariant energy functional and conformally symmetric Euler–Lagrange equations, therefore the Euler–Lagrange equations are, again, compatible with the ansatz

$$u = f(\eta) e^{in\xi + im\varphi} \tag{25}$$

using toroidal coordinates. The limit $|\mathbf{r}| \to \infty$ corresponds to the limits $\xi \to 0$ and $\eta \to 0$, therefore f has to obey $\lim_{\eta \to 0} f(\eta) = 0$ or $\lim_{\eta \to 0} f(\eta) = \infty$ in order that u be defined on $\mathbb{R}_0^3$. As u and $1/u$ are related by a symmetry transformation in both models, we may asssume for a true Hopf map that $f(\eta = 0) = 0$ without loss of generality. For u to take values in the whole target space $\mathbb{C}_0$ now requires that f takes values on the whole positive real semiaxis $\mathbb{R}_0^+$. Further, f has to take the value ∞ at $\eta = \infty$, because the pre-image of $\eta = \infty$ is a circle (the unit circle in the (x, y)-plane), whereas the preimages of finite $\eta =$ const. are surfaces (tori). So if we want the preimage of the point $u = \infty$ to be a circle, u has to take the value ∞ at $\eta = \infty$. In short, for a u within the ansatz (25) to be a genuine Hopf map, f has to obey

$$f(\eta = 0) = 0\,, \quad f(\eta = \infty) = \infty \qquad \text{or} \qquad f(t = 0) = 0\,, \quad f(t = \infty) = \infty \tag{26}$$

where $t = \sinh \eta$, as above. In Appendix A we derive the same result from an analytic expression for the Hopf index by proving that integer valuedness of the Hopf index precisely requires the boundary conditions (26).

The energy functional for the AFZ model is

$$E_{\mathrm{AFZ}}[u] = \int d^3\mathbf{r} \left(\frac{(\nabla u \cdot \nabla \bar{u})^2 - (\nabla u)^2 (\nabla \bar{u})^2}{(1 + u\bar{u})^4} \right)^{\frac{3}{4}} \tag{27}$$

and, for the ansatz (25) and using symmetric criticality,

$$E_{\mathrm{AFZ}}[f] = 4\pi^2 \int_0^\infty d\eta \sinh \eta \left(\frac{4 f^2 f_\eta^2 (n^2 + \frac{m^2}{\sinh^2 \eta})}{(1 + f^2)^4} \right)^{\frac{3}{4}} . \tag{28}$$

For reasons of simplicity we now restrict to the case $m = n$ (the discussion for $n \neq m$ is completely analogous). Further we define $F \equiv f^2$ and find for the energy functional

$$E_{\mathrm{AFZ}}[F] = 4\pi^2 m^{\frac{3}{2}} \int_0^\infty dt \frac{1 + t^2}{\sqrt{t}} \left(\frac{F_t}{(1 + F)^2} \right)^{\frac{3}{2}} \tag{29}$$

The solution of the Euler–Lagrange equation which obeys the boundary condition $F(t = 0) = 0$, $F(t = \infty) = \infty$ required for a Hopf map is $F = t^2$, see below. However, field configurations which deviate from these boundary conditions give perfectly valid, nonsingular energy densities with finite energy. E.g., for $F = c_0 + t^2$, $c_0 \geq 0$, the energy is

$$E_{\rm AFZ}(c_0) = 4\pi^2 m^{\frac{3}{2}} \int_0^\infty dt t(1+t^2)(1+c_0+t^2)^{-3} = \pi^2 (2m)^{\frac{3}{2}} \frac{2+c_0}{(1+c_0)^2}. \quad (30)$$

Exactly the same result is obtained for $F = t^2/(1+c_0 t^2)$, which deviates from the boundary condition for a Hopf map at $t = \infty$.

Next, we want to find the Euler–Lagrange equation. We introduce the function

$$G \equiv -\frac{1}{1+F} = -\frac{1}{1+f^2} \quad (31)$$

which leads to the energy functional

$$E_{\rm AFZ}[G] = 4\pi^2 m^{\frac{3}{2}} \int_0^\infty dt \frac{1+t^2}{\sqrt{t}} |G_t|^{\frac{3}{2}} \quad (32)$$

which is quite similar to the energy functional of the toy model in the preceding section. As the energy functional is homogeneous in G, we again have that the target space scale transformation

$$G \to \lambda G \quad (33)$$

is a symmetry of the Euler–Lagrange equation which does not leave invariant the energy.

Remark: as long as we only consider the energy functional (32) by itself, λ may take arbitrary values. However, the relation (31) requires, for positive semidefinite F, that $-G \in [0,1]$ and, therefore, restricts the possible values of λ to $0 \leq \lambda \leq 1$.

For the variation of the energy functional we easily find (assuming again $G_t \geq 0 \, \forall \, t$)

$$\begin{aligned} \delta E_{\rm AFZ} &\equiv E_{\rm AFZ}[G+\delta G] - E_{\rm AFZ}[G] \\ &= 6\pi^2 m^{\frac{3}{2}} \left[\frac{1+t^2}{\sqrt{t}} G_t^{\frac{1}{2}} \delta G \right]_0^\infty - \quad (34) \\ &\quad 6\pi^2 m^{\frac{3}{2}} \int_0^\infty dt \delta G \frac{d}{dt} [\frac{1+t^2}{\sqrt{t}} G_t^{\frac{1}{2}}] \quad (35) \end{aligned}$$

and therefore the Euler–Lagrange equation

$$\frac{d}{dt}[\frac{1+t^2}{\sqrt{t}}G_t^{\frac{1}{2}}]=0 \tag{36}$$

with the solution

$$\begin{aligned} G_t &= \frac{C}{2}\frac{t}{(1+t^2)^2} && (37) \\ G &= -\frac{C}{1+t^2}-D && (38) \end{aligned}$$

where C and D are non-negative real integration constants. Further,

$$F=\frac{1-C-D+t^2(1-D)}{C+D(1+t^2)} \tag{39}$$

is positive semidefinite by construction, therefore C and D are restricted to

$$C+D\leq 1 \qquad \wedge \qquad D\leq 1. \tag{40}$$

For $C=1$, $D=0$ we find the solution $F=t^2$ which obeys the boundary conditions for a Hopf map. For other values of C and D we find solutions F which do *not* obey these boundary conditions and, therefore, do not give rise to Hopf maps. Nevertheless, they are regular configurations which have finite energy. Specifically, for C strictly less than one the energy is lowered,

$$E_{\rm AFZ}=\frac{\sqrt{2}}{2}\pi^2 m^{\frac{3}{2}}C^{\frac{3}{2}}. \tag{41}$$

Again, for variations proportional to the solution, $\delta G=\epsilon G$, the boundary term (34) gives a nonzero contribution to the variation of the energy,

$$\delta E_{\rm AFZ}=3\sqrt{2}\pi^2 m^{\frac{3}{2}}\epsilon C^{\frac{3}{2}}. \tag{42}$$

Therefore, the solution to the Euler–Lagrange equation is again not a critical point of the energy functional.

At this point we may ask to which symmetry transformation corresponds the target space scale transformation $G\to\lambda G$ in terms of the original fields u and $\bar{u}$. This transformation is, in fact, already well-known (the infinitesimal version has been found in [18], and the finite version has been calculated in [26]), and it does not depend on the specific choice of toroidal coordinates, but rather is a pure target space transformation. It is based on the observation that the action density $\mathcal{L}_4$ is just the square of the pullback (under the map u) of the area twoform on the target space S^2,

$$\Omega = -i\frac{du d\bar{u}}{(1+u\bar{u})^2}. \tag{43}$$

The transformation we search is the transformation $u \to v(u,\bar{u})$ such that the area twoform is mapped to

$$\frac{du d\bar{u}}{(1+\bar{u}u)^2} \to \frac{dv d\bar{v}}{(1+\bar{v}v)^2} = \Lambda^2 \frac{du d\bar{u}}{(1+\bar{u}u)^2}. \tag{44}$$

If we introduce the real coordinates on target space $u = F^{1/2}e^{i\sigma}$ (angle and radius squared on the Euclidean plane; here we do *not* assume a specific variable dependence of the modulus squared F) and assume that $v = (\tilde{F})^{1/2}(F)e^{i\sigma}$ (i.e. u and v have the same argument, and the transformation only affects the modulus) then we get the equation

$$\frac{\tilde{F}'(F)dFd\sigma}{(1+\tilde{F})^2} = \Lambda^2 \frac{dFd\sigma}{(1+F)^2} \tag{45}$$

or

$$\frac{\tilde{F}'}{(1+\tilde{F})^2} = \frac{\Lambda^2}{(1+F)^2} \tag{46}$$

with the solution

$$\frac{1}{1+\tilde{F}} = \frac{\Lambda^2}{1+F} + c \tag{47}$$

where c is a constant of integration. For $c = 0$ we just get $\tilde{G} = \Lambda^2 G$, that is, the scale transformation (33) with $\lambda = \Lambda^2$. If, instead, we impose the boundary condition that $v = 0$ for $u = 0$ then $c = 1 - \Lambda^2$ and the solution is

$$\tilde{F} = \frac{\Lambda^2 F}{\Lambda^2 + (1+F)(1-\Lambda^2)} \tag{48}$$

or

$$v = \frac{\Lambda u}{[\Lambda^2 + (1+\bar{u}u)(1-\Lambda^2)]^{1/2}}. \tag{49}$$

Observe that Λ, again, is restricted to $\Lambda \leq 1$ if u is a genuine Hopf map which covers the whole target space. Further, v is no longer a true Hopf map for $\Lambda < 1$.

Remark: the transformation (49) is a target space symmetry transformation of the field equations, i.e., a target space transformation which maps solutions u into solutions v. It is, however, not a symmetry of the action, but, instead, rescales the action density $\mathcal{L}_4$ by $\mathcal{L}_4 \to \Lambda^4 \mathcal{L}_4$. It is, therefore, not a Noether symmetry and does not define a Noether current and conserved charge. On the other hand, a combination of the target space transformation (49) and the base space scale transformation (dilatation) $(t, \mathbf{r}) \to (\mu t, \mu \mathbf{r})$, $\mu = \Lambda^{-4}$, leaves the action invariant and, therefore, gives rise to a Noether current and conserved charge, see [18] for details.

The transformation (49) is a pure target space transformation and, therefore, the rescaling of the action density $\mathcal{L}_4$ by $\mathcal{L}_4 \to \Lambda^4 \mathcal{L}_4$ is independent of the base space. As a consequence, stability problems which are similar to the ones discussed here can be expected for all lagrangian densities which are homogeneous in $\mathcal{L}_4$, for arbitrary base spaces. This issue will be discussed further in the last section.

2.3 The Nicole model

Here we just want to demonstrate briefly that, at least for field configurations belonging to the ansatz (25), soliton solutions of the Nicole model with non-zero Hopf index are indeed separated by an infinite energy barrier from the trivial sector, and that field configurations which do not obey the boundary conditions necessary for an integer Hopf index have infinite energy. The energy functional of the Nicole model is just a non-integer power of the CP^1 model Lagrangian,

$$E_{\mathrm{Ni}}[u, \bar{u}] = \int d^3\mathbf{r} \left(\frac{\nabla u \cdot \nabla \bar{u}}{(1 + u\bar{u})^2} \right)^{\frac{3}{2}} . \tag{50}$$

For the ansatz (25), and for $m = n$, which again we assume for simplicity, this reduces to

$$E_{\mathrm{Ni}}[f] = \int_0^\infty dt t (1 + t^2)(1 + f^2)^{-3} \left(f_t^2 + m^2 \frac{f^2}{t^2} \right)^{\frac{3}{2}} \equiv \int_0^\infty dt \mathcal{E}(t). \tag{51}$$

Now let us assume that $f(0) = c_0 > 0$. For the density $\mathcal{E}(t)$ this implies

$$\lim_{t\to 0} \mathcal{E}(t) \simeq \lim_{t\to 0} t(1+c_0^2)^{-3} \left(f_t^2 + m^2 \frac{c_0^2}{t^2} \right)^{\frac{3}{2}} \geq t^{-2} \frac{m^3 c_0^3}{(1+c_0^2)^3} \tag{52}$$

which has a nonintegrable singularity at $t = 0$. Equivalently, in the limit $t \to \infty$ we have, assuming $f(\infty) = c_\infty$,

$$\lim_{t\to \infty} \mathcal{E}(t) \simeq t^3(1+c_\infty^2)^{-3} \left(f_t^2 + m^2 \frac{c_\infty^2}{t^2} \right)^{\frac{3}{2}} \geq \frac{m^3 c_0^3}{(1+c_0^2)^3}. \tag{53}$$

Therefore, the density is bounded from below by a nonzero constant, and its integral again will be infinite.

It follows that for field configurations which obey boundary conditions such that the corresponding Hopf index is non-integer, the energy in the Nicole model automatically is infinite. In other words, it is a necessary condition for a finite energy field configuration to have integer Hopf index. The condition is not sufficient, however. For fields with the leading behaviour $\lim_{t\to 0} \mathcal{E}(t) \simeq t^{\alpha_0}$, $\lim_{t\to\infty} \mathcal{E}(t) \simeq t^{\alpha_\infty}$, for instance, it may be checked easily that finite energy requires $\alpha_0 > 1/3$ and $\alpha_\infty > 1/3$, which is more restrictive than just $f(0) = 0$ and $f(\infty) = \infty$.

3 Full energy functional and nonzero fluxes

3.1 Toy model

The variation of the energy functional of the toy model for an arbitrary scalar field is

$$\begin{aligned} \delta E &\equiv E[\Phi + \delta\Phi] - E[\Phi] \\ &= 3 \int d^3\mathbf{r} \nabla \cdot (\delta\Phi |\nabla\Phi| \nabla\Phi) \qquad (54) \\ &- 3 \int d^3\mathbf{r} \delta\Phi \nabla \cdot (|\nabla\Phi|\nabla\Phi) \qquad (55) \end{aligned}$$

and the resulting Euler–Lagrange equation is just the conservation equation

$$\nabla \cdot (|\nabla\Phi|\nabla\Phi) \equiv \nabla \cdot \mathbf{J} = 0. \tag{56}$$

The nonzero contribution of the solution (17) to the boundary terms of the variation δE of the reduced energy functional indicates the presence of nonzero delta-function like source terms in the conservation equation (56). Further, these source terms we expect to be concentrated at the loci of the boundaries of the reduced system, that is, at $\eta = 0$ (the z axis) and at $\eta = \infty$ (the unit circle in the (x, y)-plane). This we want to investigate in the sequel. For the solution $\Phi(\eta)$ of the last section with $\Phi_\eta = C \sinh^{-\frac{1}{2}} \eta$ the current $\mathbf{J}$ is

$$\mathbf{J} = q^2 \Phi_\eta^2 \hat{e}_\eta = C^2 \frac{q^2}{\sinh \eta} \hat{e}_\eta. \tag{57}$$

We now want to calculate the flux of this current through a torus $\eta =$ const. The surface element of a torus $\eta =$ const. is just

$$d\mathbf{\Sigma}_{\mathrm{T}} = \frac{\sinh \eta}{q^2} \hat{e}_\eta d\xi d\varphi \tag{58}$$

therefore the flux simply is

$$\int d\mathbf{\Sigma}_{\mathrm{T}} \cdot \mathbf{J} = \int_0^{2\pi} d\xi \int_0^{2\pi} d\varphi C^2 = 4\pi^2 C^2. \tag{59}$$

So there is some nonzero total flux emerging from the z axis and streaming towards the unit circle

$$\mathcal{C} = \{\vec{x} \in I\!R^3 : \ z = 0 \ \wedge \ r^2 = 1\}. \tag{60}$$

The total flux escaping to infinity is zero, as may be checked by explicit calculation. However, this already follows from the fact that the total charges distributed along the z axis and along the unit circle $\mathcal{C}$ are equal in magnitude and opposite in sign. The line charge density along the circle $\mathcal{C}$ is constant, as is obvious for symmetry reasons. The line charge density along the z axis may be calculated by calculating the flux through a cylinder around the z axis with infinitesimally small radius. For infinitesimally small radius, the top and bottom of the cylinder do not contribute. The surface element of the cylinder mantle in cylinder coordinates (ρ, φ, z), $\rho^2 = x^2 + y^2$, is

$$d\mathbf{\Sigma}_{\mathrm{M}} = \rho \hat{e}_\rho d\varphi dz \tag{61}$$

Further, in the limit $\rho \to 0$, $\hat{e}_\eta \simeq \hat{e}_\rho$, and therefore

$$\begin{aligned} \int d\mathbf{\Sigma}_{\mathrm{M}} \cdot \mathbf{J} &= \lim_{\rho \to 0} C^2 \int_{z_1}^{z_2} dz \int_0^{2\pi} d\varphi \rho \frac{q^2}{\sinh \eta} \\ &= 2\pi C^2 \int_{z_1}^{z_2} dz \frac{2}{1+z^2} \end{aligned} \tag{62}$$

where we used

$$\rho^2 = \frac{\sinh^2 \eta}{q^2} \tag{63}$$

and

$$\sinh^2 \eta = \frac{4\rho^2}{4z^2 + (\rho^2 + z^2 - 1)^2} \quad \Rightarrow \quad \lim_{\rho \to 0} \frac{\sinh \eta}{\rho} = \frac{2}{1+z^2}. \tag{64}$$

Consequently, the solution $\Phi(\eta)$ solves in fact the inhomogeneous equation

$$\nabla \cdot \mathbf{J} = 4\pi C^2 \mathcal{Q} \equiv 4\pi C^2 \left(\delta(x)\delta(y) \frac{1}{1+z^2} - \delta(z)\delta(\rho^2 - 1) \right), \tag{65}$$

and the total charge of the density distribution $\mathcal{Q}$ is zero,

$$\int d^3\mathrm{r} 4\pi C^2 \mathcal{Q} = \int d^3\mathrm{r} \nabla \cdot \mathbf{J} = 0, \tag{66}$$

as may be checked easily.

Remark: The observation that the toy model is equivalent to the purely electrostatic sector of a nonlinear theory of electrodynamics may add some additional interest to it. Indeed, take the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \left(\frac{F_{\alpha\beta} F^{\alpha\beta}}{2\Omega^4} \right)^{\frac{1}{2}} \tag{67}$$

where Ω is a constant with the dimension of mass, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A_\mu = (V, \vec{A})$. This leads to the field equation (Gauss law)

$$\partial_\nu \left[F^{\mu\nu} \left(\frac{F_{\alpha\beta} F^{\alpha\beta}}{2\Omega^4} \right)^{\frac{1}{2}} \right] = 0 \tag{68}$$

or, in the electrostatic case ($\vec{A} = 0$, $V = V(\mathbf{r})$, $\vec{E} = -\nabla V$), to

$$\nabla \cdot (|\nabla V| \nabla V) = 0 \tag{69}$$

which is identical to Eq. (56) when the potential V is identified with the scalar field Φ of the toy model. Therefore, we have the electrostatic solutions

$$V_t = \frac{C}{\sqrt{t(1+t^2)}} \tag{70}$$

and

$$\vec{E} = -qV_\eta \hat{e}_\eta \equiv -q\sqrt{1+t^2}V_t\hat{e}_\eta = -q\frac{C}{\sqrt{t}}\hat{e}_\eta. \tag{71}$$

The electric induction in this model

$$\vec{D} = \frac{3q^2}{2\Omega^2}|\vec{E}|\vec{E} \tag{72}$$

is proportional to the current $\mathbf{J}$ of the toy model and, therefore, obeys the inhomogeneous divergence equation

$$\nabla \cdot \vec{D} = -\frac{3C^2}{2\Omega^2}\mathcal{Q} \tag{73}$$

where the singular line charge density $\mathcal{Q}$ is defined in Eq. (65).

3.2 AFZ model

The variation of the energy functional of the AFZ model is

$$\begin{aligned} \delta E_{\mathrm{AFZ}} &\equiv E[u, \bar{u} + \delta\bar{u}] - E[u, \bar{u}] \\ &= \frac{3}{2}\int d^3\mathbf{r}\nabla \cdot \left(\delta\bar{u}(1+u\bar{u})^{-2}\mathbf{L}\right) \qquad (74) \\ &- \frac{3}{2}\int d^3\mathbf{r}\delta\bar{u}(1+u\bar{u})^{-2}\nabla \cdot \mathbf{L} \qquad (75) \end{aligned}$$

where

$$\mathbf{L} \equiv (1+u\bar{u})^{-1}H^{-\frac{1}{4}}\mathbf{K} \tag{76}$$

$$H \equiv (\nabla u \cdot \nabla \bar{u})^2 - (\nabla u)^2(\nabla \bar{u})^2 \tag{77}$$

$$\mathbf{K} \equiv (\nabla u \cdot \nabla \bar{u})\nabla u - (\nabla u)^2\nabla \bar{u}. \tag{78}$$

The resulting Euler–Lagrange equation is the conservation equation

$$\nabla \cdot \mathbf{L} = 0. \tag{79}$$

We want to investigate whether, again, there are delta-function like source terms present at the r.h.s. of this conservation equation. However, for the AFZ model things are complicated by the fact that, due to the infinitely many symmetries of the model, there exist in fact infinitely many currents such that their conservation is equivalent to the Euler–Lagrange equations. Indeed, with the help of the identity

$$\nabla u \cdot \mathbf{K} \equiv 0 \tag{80}$$

it easily follows that together with the conservation equation (79) we have the infinitely many conservation equations

$$\nabla \cdot \mathbf{L}^{\zeta} \equiv \nabla \cdot \zeta \mathbf{L} = 0 \tag{81}$$

where $\zeta = \zeta(u)$ is an arbitrary function of u only. The currents $\mathbf{L}^{\zeta}$ are, in general, complex, but it is easy to construct an equivalent set of real conserved currents,

$$\mathbf{L}^{\mathcal{G}} \equiv -i \left(\mathcal{G}_u \mathbf{L} - \mathcal{G}_{\bar{u}} \bar{\mathbf{L}} \right) \tag{82}$$

where $\mathcal{G}_u \equiv \partial_u \mathcal{G}$, etc., and $\mathcal{G} = \mathcal{G}(u, \bar{u})$ is an arbitrary *real* function of u and $\bar{u}$. The real currents $\mathbf{L}^{\mathcal{G}}$ are just the Noether currents of the area-preserving target space diffeomorphisms, which are wellknown symmetries of the AFZ model [18, 27].

For the ansatz (25) for u in toroidal coordinates (and $m = n$), the current (76) is

$$\mathbf{L} = 2\frac{q^2}{1+f^2} e^{im(\xi+\varphi)} \left[\left(\frac{m}{\tanh \eta} f \right)^{\frac{3}{2}} f_\eta^{\frac{1}{2}} \hat{e}_\eta + i \left(\frac{m}{\tanh \eta} f \right)^{\frac{1}{2}} f_\eta^{\frac{3}{2}} \left(\hat{e}_\xi + \frac{\hat{e}_\varphi}{\sinh \eta} \right) \right]. \tag{83}$$

This current will give a zero total flux through a torus $\eta =$ const. because of the angular factor $\exp[im(\xi + \varphi)]$. By inspection, it is obvious that a nonzero flux will exist for the current $\mathbf{L}^{\zeta}$ with $\zeta = u^{-1}$, because then the angular factor is absent. Multiplying by the surface element of a torus $\eta =$ const. we find

$$\mathbf{L}^{\frac{1}{u}} \cdot d\mathbf{\Sigma}_{\mathrm{T}} \quad = \quad 2m^{\frac{3}{2}} \frac{\cosh^{\frac{3}{2}}}{\sinh^{\frac{1}{2}} \eta} \left(\frac{f f_\eta}{(1+f^2)^2} \right)^{\frac{1}{2}}$$

$$\begin{aligned} &= \sqrt{2} m^{\frac{3}{2}} \frac{1+t^2}{\sqrt{t}} \left(\frac{F_t}{(1+F)^2} \right)^{\frac{1}{2}} \\ &= C^{\frac{1}{2}} m^{\frac{3}{2}} \end{aligned} \tag{84}$$

where we used the solution

$$\frac{F_t}{(1+F)^2} \equiv G_t = \frac{C}{2} \frac{t}{(1+t^2)^2} \ , \tag{85}$$

and therefore the flux $4\pi^2 m^{\frac{3}{2}} \sqrt{C}$ through an arbitrary torus of $\eta = \text{const}$.

Remark: the current with nonzero flux may be equally found among the real currents $\mathbf{L}^{\mathcal{G}}$ for the choice

$$\mathcal{G} \equiv \mathcal{G}^{\mathrm{f}} = \frac{i}{2}(\ln u - \ln \bar{u}). \tag{86}$$

The current $\mathbf{L}^{\mathcal{G}^{\mathrm{f}}}$ is identical to the current $\mathbf{J}$ of the toy model and, therefore, obeys the inhomogeneous divergence equation

$$\nabla \cdot \mathbf{L}^{\mathcal{G}^{\mathrm{f}}} = 4\pi C^{\frac{1}{2}} m^{\frac{3}{2}} \left(\delta(x)\delta(y) \frac{1}{1+z^2} - \delta(z)\delta(\rho^2 - 1) \right). \tag{87}$$

4 Discussion

The main result of this paper is the observation that in the AFZ model for any solution u there exists a one-parameter family of solutions $v(u, \Lambda)$ which connects the given solution u at $\Lambda = 1$ to the vacuum solution $v(u, 0) = 0$, see Eq. (49). Therefore, sectors of field configurations with different topological (Hopf) index are not separated by an infinite energy barrier, and there exist well-defined (i.e. single-valued, nonsingular) field configurations with finite energy in the model which do not correspond to integer Hopf index. For static solutions (i.e., solitons), this observation immediately implies some problems. The first problem is that the mere existence of soliton solutions under these conditions poses an apparent paradox. Indeed, the static field equations which the solitons obey are just the Euler-Lagrange equations for the variational problem of the energy functional $E_{\mathrm{AFZ}}[u, \bar{u}]$ and, therefore, if u_0 is a soliton solution, it should hold that

$$\left[\frac{d}{d\epsilon} E_{\mathrm{AFZ}}[u_0, \bar{u}_0 + \epsilon \delta \bar{u}] \right]_{\epsilon=0} = \left[\frac{d}{d\epsilon} E_{\mathrm{AFZ}}[u_0 + \epsilon \delta u, \bar{u}_0] \right]_{\epsilon=0} = 0 \tag{88}$$

for arbitrary δu. But in our case this is not true. We have that $E_{\mathrm{AFZ}}[v,\bar{v}] = \Lambda^3 E_{\mathrm{AFZ}}[u,\bar{u}]$ and therefore

$$\left[\frac{d}{d\epsilon}E_{\mathrm{AFZ}}[u_0,\bar{u}_0+\epsilon\delta\bar{u}]\right]_{\epsilon=0}+\left[\frac{d}{d\epsilon}E_{\mathrm{AFZ}}[u_0+\epsilon\delta u,\bar{u}_0]\right]_{\epsilon=0}=3E_{\mathrm{AFZ}}[u_0,\bar{u}_0] \quad (89)$$

for $\Lambda = 1-\epsilon$ and $\delta u = u_0(1+u_0\bar{u}_0)$. The resolution of this apparent paradox can be understood in different ways. For the reduced energy functional within the ansatz in toroidal coordinates (see Section 2), it turned out that in addition to the bulk contribution to the variation of the energy functional (which is zero for a solution) there are boundary contributions, which are nonzero for certain variations, even when the variations are performed about a soliton solution.

For the full energy functionals the nonzero contributions to the variation about a solution cannot be attributed to boundary terms, because there are no boundaries in the base space $\mathbb{R}^3$, and the soliton solutions are behaving well in the limit $\mathbf{r}\to\infty$. For the toy model, the resolution of the paradox is quite simple. The static field equation corresponds to a conservation equation $\nabla\cdot\mathbf{J}=0$ for some current $\mathbf{J}$, whereas the "solution" obeys, in fact, an inhomogeneous conservation equation instead, with some delta-function like sources and sinks on the r.h.s. For the AFZ model, the resolution is slightly more subtle. The static field equation is again equivalent to a conservation equation $\nabla\cdot\mathbf{L}=0$, but now the well-known soliton solutions do not induce an inhomogeneous source term in this conservation equation. However, due to the infinitely many symmetries of the AFZ model, there formally exist infinitely many more conservation equations $\nabla\cdot\mathbf{L}^\zeta=0$ for soliton solutions. It turns out that, for nontrivial soliton solutions, one of these infinitely many conservation equations is, in fact, inhomogeneous and contains, again, a singular flux term on the r.h.s., see Eq. (87). Further, the strength of the delta-function like inhomogeneous term is not varied in the variational derivation of the field equations. Therefore, the value of Λ, which plays the role of an integration constant for solutions within the ansatz in toroidal coordinates ($\Lambda=\sqrt{C}$, see Eqs. (85) and (87)), remains fixed, which explains the existence of solutions to the variational equations.

The next issue to be discussed is, of course, the stability of solutions (that is, the stability under small perturbations). Naivly, one might assume that the presence of the inhomogeneous term in one of the conservation equations stabilizes the solutions by fixing the value of Λ. However, this might not be correct for the following reasons. Firstly, the inhomogeneous term does not correspond to a nonzero charge, because the spatial integral of the inhomogeneity is zero, see Eq. (66). Secondly, the fact that the inhomogeneous term is not varied by the variation of the energy functional does not imply that it is conserved under dynamical evolution of the system. The conservation under dynamical evolution would require that Λ (or a related quantity, like, e.g., the Hopf charge, which varies under the transformation (49)) may be expressed by one or more of the conserved charges of the theory. For static solutions this most likely does not happen, however, because most of the conserved charges are trivially zero for static solutions (e.g., all the infinitely many conserved charges related to the infinitely many area-preserving target space diffeomorphisms contain time derivatives and are, therefore, zero for static solutions). We conclude that the stability of the soliton solutions in the AFZ model is problematic.

Although the explicit calculations have been done for the static AFZ model with Lagrangian $(\mathcal{L}_4)^{\frac{3}{4}}$ and base space $\mathbb{R}^3$, some conclusions may be drawn for more general models based on the Lagrangian $\mathcal{L}_4$ and with different base spaces. For instance, static soliton solutions have been found in [28] for the Lagrangian $\mathcal{L}_4$ and the three-sphere S^3 as base space (that is, space-time $S^3 \times \mathbb{R}$). These static solutions solve the Euler–Lagrange equations which follow from varying the static energy functional

$$E = \int_{S^3} dV_{S^3} \frac{(\nabla u \cdot \nabla \bar{u})^2 - (\nabla u)^2 (\nabla \bar{u})^2}{(1 + u\bar{u})^4} \tag{90}$$

(where dV_{S^3} is the volume element on S^3) and have finite energy, therefore the same arguments as above apply. Due to the symmetry (49) these solutions are connected to the vacuum and, in addition, are not critical points of the energy functional. Therefore, the mere existence of these solutions requires the presence of inhomogeneous singular flux terms in some (at least one) of the infinitely many conservation equations. This line of arguments remains valid for static finite energy solutions for Lagrangians which are arbitrary powers of the Lagrangian $\mathcal{L}_4$, and for arbitrary base spaces.

The situation is slightly different for time-dependent solutions. Indeed, time dependent solutions for the Lagrangian $\mathcal{L}_4$ have been found both for base space-time $\mathbb{R}^4$ and $S^3 \times \mathbb{R}$ in [29] and [30], respectively. These solutions have integer Hopf charge and are particle-like, that is, they have finite energy and infinite action. These solutions are again connected to the vacuum by the symmetry transformation (49), but the apparent paradox discussed above is absent for these time dependent solutions. The energy transforms like $E \to \Lambda^4 E$ under the transformation (49), but the time dependent field equations are not derived by varying the energy functional, and time dependent solutions are not required to be critical points of the energy functional. The action I transforms like the energy under the symmetry transformation (49), $I \to \Lambda^4 I$, so the same apparent paradox would exist for time dependent finite action solutions ("instantons"). But it does not exist for time dependent solutions with infinite action (particle-like solutions). Therefore, the existence of inhomogeneous singular flux terms in some conservation equations is not required by the existence of these time dependent solutions (although we do not know whether such inhomogeneous singular flux terms do or do not show up for the time dependent solutions). Even the issue of stability may be less problematic for these time dependent solutions. The point is that the infinitely many conserved Noether charges, which exist in the model as a consequence of its infinitely many symmetries, are nontrivial (i.e., nonzero) for time dependent solutions. It is quite possible that these infinitely many conservation laws stabilize the time dependent solutions in spite of the fact that those solutions are connected to the vacuum by a symmetry transformation.

Another interesting feature of models based on the Lagrangian density $\mathcal{L}_4$, which is related to the existence of finite energy configurations with non-integer Hopf index, is the rich vacuum structure of such models. Indeed, a complex field $u(h)$ which only depends on one real function $h(t, \mathbf{r})$ gives zero when inserted into the Lagrangian density $\mathcal{L}_4$, implying that the vacuum manifold is infinite dimensional. This may be understood in geometric terms by the observation that the quartic Lagrangian $\mathcal{L}_4$ (and the related energy density) is just the square of the pullback (under the map u) of the area two-form on target space S^2. A field $u(h)$ of the above type is a map $\mathbb{R}^3 \to \mathcal{M}^1 \to S^2$, where $\mathcal{M}^1$ is a one-dimensional manifold, and the pullback of a two-form onto a one-dimensional manifold is zero. As a consequence, static fields u which have a functional dependence $u(h)$ in the limit $\mathbf{r} \to \infty$ will have finite energy. In the AFZ model and within the ansatz of toroidal coordinates, for instance, it happens that $\lim_{\mathbf{r}\to\infty} u = c_0 \exp(im\varphi)$ whenever $f(\eta)$ obeys the boundary condition $f(0) = c_0$.

The rigorous results, as well as the detailed arguments summarized in this section for some nonlinear models in 4 dimensions, are the main contribution of our work. These models are special, but they are related to and share many properties of some relevant fundamental and effective field theories. The Lagrangian $\mathcal{L}_4$, for instance, is just the quartic part of the Faddeev–Niemi model. The stability of this term has been investigated recently in [31], together with some generalizations (Lagrangians based on the Kaehler two-form on various target spaces). However, the stability analysis of the quartic Lagrangian $\mathcal{L}_4$ in [31] focused on the quadratic variation of the energy functional, whereas we found that already the linear variation is problematic. Our general conclusion, therefore, is that the stability of the extended solutions, which is often taken more or less for granted, is an extremely subtle and difficult issue which, as shown here, can be understood in detail by a careful analysis in some cases. Our methods can certainly be useful for that purpose. Note also that in numerical analysis the restriction to a fixed topological sector is often imposed from the outset by assuming the corresponding boundary conditions. Under these conditions, it would not be possible via numerical analysis to detect stability problems of the type discussed here.

Appendix A

Here we want to demonstrate briefly from the analytic expression for the Hopf index that for fields u within the toroidal ansatz (25) the condition that u has integer Hopf index precisely requires that f obeys the boundary condition (26). The analytic expression for the Hopf index is

$$Q = \frac{1}{16\pi^2} \int d^3 r \vec{\mathcal{A}} \cdot \vec{\mathcal{B}} \tag{91}$$

where $\vec{\mathcal{B}}$ is the Hopf curvature

$$\vec{\mathcal{B}} = \frac{2}{i} \frac{\nabla u \times \nabla \bar{u}}{(1 + u\bar{u})^2} \tag{92}$$

and $\vec{\mathcal{A}}$ is the gauge potential for the "magnetic field" $\vec{\mathcal{B}}$, $\vec{\mathcal{B}} = \nabla \times \vec{\mathcal{A}}$. There is no local expression for $\vec{\mathcal{A}}$ in terms of the Hopf map u alone, but when u is expressed with the help of a four component unit vector field e_α, $\alpha = 1, \ldots, 4$ like

$$u = \frac{e_1 + ie_2}{e_3 + ie_4}, \tag{93}$$

then an explicit, local expression for the gauge potential $\vec{\mathcal{A}}$ in terms of the e_α exists and is given by

$$\vec{\mathcal{A}} = \frac{2}{i}[(e_1 - ie_2)\nabla(e_1 + ie_2) + (e_3 - ie_4)\nabla(e_3 + ie_4)]. \tag{94}$$

In our case we may choose

$$e_1 + ie_2 = \frac{f}{\sqrt{1+f^2}}e^{im\varphi}, \quad e_3 + ie_4 = \frac{1}{\sqrt{1+f^2}}e^{-in\xi} \tag{95}$$

and, therefore,

$$\vec{\mathcal{A}} = \frac{2}{i}q\frac{1}{1+f^2}\left(\frac{2ff_\eta}{1+f^2}\hat{e}_\eta + \frac{imf^2}{\sinh\eta}\hat{e}_\varphi - in\hat{e}_\xi\right) \tag{96}$$

$$\vec{\mathcal{B}} = 4q^2\frac{ff_\eta}{(1+f^2)^2}\left(n\hat{e}_\varphi - \frac{m}{\sinh\eta}\hat{e}_\xi\right) \tag{97}$$

and

$$Q = \frac{1}{16\pi^2}\int d\eta d\xi d\varphi 8mn\frac{ff_\eta}{(1+f^2)^2} = mn\left[-\frac{1}{1+F(\eta)}\right]_0^\infty \tag{98}$$

(remember $F \equiv f^2$). It follows that the conditions for integer, nonzero Hopf index are precisely $f(0) = 0$ and $f(\infty) = \infty$ (or the inverse conditions $f(0) = \infty$ and $f(\infty) = 0$; however, as $u \to (1/u)$ is a symmetry transformation for both models, these cases are equivalent).

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